ARTICLE NO. 79-4087

Traveling Between the Lagrange Points and the Moon

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A detailed study is presented of the possible free-fall trajectories from the Moon to the four Lagrangian points L_1, L_2, L_4, L_5, and back from these points to the Moon. The Lagrangian point L_3 has not been included in the study because it is considered of less practical interest in connection with travel from and to an eventual large space station or space colony. The problem of compromising between short transit times and small residual velocity at L_1, which is proportional to the amount of fuel needed, is considered in detail. Several families of possible trajectories for each of the four Lagrangian points are represented. For L_1 there is a family of slow indirect trajectories and a family containing both fast and slow direct trajectories. For L_2 we found four different families, three of which have a slow and a fast branch. As for the triangular libration points, L_4 and L_5, we describe a family of transfer orbits for each of them. Each family also has a slow and a fast branch. Finally, some relations between transfer orbits and the classical families of periodic solutions of the restricted three-body problem are discussed.

I. Introduction

IN the past two years some interest has developed in the construction of large manned space stations. These stations would be orbiting in the Earth-Moon system and would be constructed and assembled in space from materials taken partly from the Earth but mostly from the Moon. Several transportation systems have been considered. They range from magnetic accelerators located on the Moon, to space shuttle type vehicles traveling back and forth. Several possible locations for the large manned station have also been proposed, and the equilibrium points (Lagrange points) are favorite candidates, especially the triangular points, L_4 (60 deg ahead of the Moon) and L_5 (60 deg behind the Moon), or even at the collinear points, L_1 (in front of the Moon) and L_2 (behind the Moon) (see Refs. 1-4).

Thus, we have undertaken a systematic study of the possible free-fall trajectories connecting the Moon with any of the equilibrium points L_1, L_2, L_3, and L_4. Only those free-fall trajectories that are of sufficient practical value are considered here – the trajectories should be as direct as possible, with short transit times and small arrival velocity at the Lagrangian point. As a model, we have taken, for the present initial exploration, the well-known circular-restricted three-body problem in two dimensions. The Earth and the Moon are the only two acting bodies; they are point-masses in circular orbits around each other. A Levi-Civita regularization was used for the numerical integration of trajectories in proximity of the Moon.[5]

In this article we do not consider the important navigation and guidance aspects associated with traveling in Earth-Moon space, nor are we considering the problem of stationkeeping at the Lagrangian points. Some of these problems have already received much attention in the literature in relation to the use of these points for data relays and communication purposes.[6-13] Also, we do not study the problem of the attitude stability of a space station at the Lagrangian points. This problem too has received attention by several researchers.[14]

The stabilization of the three collinear Lagrange points by means of an active control has also been proposed and investigated by Paul and Shapiro.[15] Of course, much work has also been done on the pure celestial mechanics properties,[5,16-19] and on the astronomical implications of the triangular points and the Trojan asteroids.[20-22]

One of the most extensive studies of the use of the libration points for satellite locations was made by Farquhar.[7] He considers the stationkeeping and control of the satellite, as well as the possible scientific applications. These aspects and applications of the equilibrium points are not considered in this paper.

Section II of this paper reviews the general characteristic of the orbits in the Earth-Moon system. Section III discusses the mirror image theorem, which allows us to study only trajectories *to the Moon*. Return trajectories *from the Moon* follow automatically by application of a symmetry theorem.[23] Finally, in the last few sections, we describe the most important numerical results that were obtained. As was said before, we only concentrate on free-fall trajectories with a low velocity at the libration points (about 200 m/s at the most). These trajectories could be used for travel to and from these points with fairly low fuel consumption or energy expenditure. For a space shuttle type vehicle (in the text we interchangeably use the words satellite, spacecraft, space station, or space vehicle), the only fuel consumption would be to reach some escape velocity of 100-200 m/s at the libration point and then for approach and landing on the Moon (or insertion into a lunar parking orbit).

As future studies, we would recommend to examine the sensitivity of our trajectories (partial derivatives of final parameters with respect to initial conditions). We would also recommend a study of the solar perturbations on the present orbits. It was recently shown by Schutz[24,25] that the solar perturbations are a serious problem in stationkeeping at a Lagrangian point.

II. The Earth-Moon System

The model for the present study will be the well-known planar circular-restricted three-body problem. In this problem one of the three objects, say m_3, is supposed to be of negligible mass compared with the two others, m_1 and m_2. In the present context, m_1 will be the Earth, m_2 the Moon, and m_3 the spacecraft or space station. The motion of m_1 and m_2 around their center of mass is supposed to be the circular Keplerian motion. The motion of m_3 is then obtained by solving the equations of motion numerically in terms of the

Received June 5, 1978; revision received Jan. 10, 1979.

Index categories: Satellite Communication Systems (including Terrestrial Stations); Lunar and Interplanetary Trajectories; Space Station Systems, Manned.

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initial conditions. As was said above, we assume that m_3 remains in the plane of the motion of the two primaries, m_1 and m_2.

The present work corresponds to the Earth-Moon mass ratio, with value

$$m_1/m_2 = 398601.2/4902.7835 = 81.301$$

Representing the masses of the Earth and the Moon, respectively, by $1-\mu$ and μ gives us the value of $\mu = 1/82.301 = 0.0121505206$ used in this study.

The unit of length is taken equal to the average Earth-Moon distance (384,400 km). We take the unit of time in such a way that the period of m_1 and m_2 in their circular motion is exactly 2π (instead of the real value of about 28 days, or better, 2,360,590 s). This system of units, together with the convention that the universal gravitation constant is +1 and the sum $m_1 + m_2$ also +1 forms the classical system of canonical units. In this system of units, a canonical unit of time is thus about 375,700 s, or 4.348 days, or 104 h. From the above factors for converting length and time, we also derive the velocity conversion factor. The canonical units of velocity should be multiplied by 384400/375700 = 1.023 in order to obtain km/s. For all practical purposes we may thus say that canonical units of velocity are the same as km/s.

With the above units, and recalling that we take as origin of coordinates the center of mass of the whole system, the Earth may be assumed to describe the circular path

$$\xi_1 = -\mu \cos t \qquad \eta_1 = -\mu \sin t$$

while the Moon is on the other circular orbit

$$\xi_2 = (1-\mu)\cos t \qquad \eta_2 = (1-\mu)\sin t$$

with radius $1-\mu = 0.9878494794$. The differential equations for the motion of the satellite are then

$$\frac{d^2\xi}{dt^2} = -\frac{\partial V}{\partial \xi} \qquad \frac{d^2\eta}{dt^2} = -\frac{\partial V}{\partial \eta}$$

where V is the potential energy

$$V = \frac{1-\mu}{\mu_1} - \frac{\mu}{\mu_2}$$

They derive from the Lagrangian

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - V$$

where overdots indicate time derivatives.

The inertial coordinates which have just been introduced are sometimes called the "sidereal" coordinates. However, it is much more convenient to use the "synodical" coordinate system (xy), which rotates around the center of mass with unit angular velocity. We convert from the sidereal system to the synodical system with the rotation equations:

$$\xi = x \cos t - y \sin t$$

$$\eta = x \sin t + y \cos t$$

In the rotating system, the Earth is at $x_1 = -\mu$ and the Moon at $x_2 = 1-\mu$, both permanently on the x axis.

The Lagrangian of the motion can now be written as

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + (x\dot{y} - y\dot{x}) - \tfrac{1}{2}(x^2 + y^2) - V$$

We recognize the familiar Coriolis terms, linear in the velocity components $(\dot{x},\dot{y})$, which result in an apparent acceleration perpendicular and proportional to the velocity. This is also apparent in the new equations of motion

$$\ddot{x} - 2\dot{y} = x - \frac{\partial V}{\partial x} = \frac{\partial \Omega}{\partial x}$$

$$\ddot{y} + 2\dot{x} = y - \frac{\partial V}{\partial y} = \frac{\partial \Omega}{\partial y}$$

where

$$\Omega = \tfrac{1}{2}(x^2 + y^2) + \frac{1-\mu}{\mu_1} + \frac{\mu}{\mu_2}$$

Among the many advantages of the rotating coordinate system, the most important is probably that it makes the Lagrangian autonomous. This results in the existence of the energy integral (Jacobi intergral):

$$E = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \Omega(x,y) = \text{const}$$

Not only can this integral be used as a simple verification of the precision of the numerical integration, but it also contains useful information relating to the possible regions of motion and the existence of the five Lagrange points (equilibrium points).

The curves

$$E_0 = -\Omega(x,y)$$

are called zero-velocity curves, or equipotential curves, with energy E_0, while the five critical points of the function $\Omega(x,y)$, known as L_1, L_2, L_3, L_4, L_5 (Lagrange points) are known to be equilibrium solutions of the equations of motion. The exact form of the equipotential curves and the locations of the equilibrium points can be seen in Ref. 19 (pp. 8-9) or Ref. 5 (p. 185).

The coordinates of the five equilibrium points, together with the corresponding value of the potential energy $E_0 = -\Omega$, are given in Table 1.

III. The Mirror Image Theorem

We will first mention here an important property of the equations of motion in rotating coordinates which allows us to relate the return trajectories to the Moon with the direct trajectories from the Moon without actually having to perform the supplementary numerical calculations. In the rotating system all solutions form symmetric pairs with respect to the syzygy axis (the Earth-Moon line).[23]

As a result of this symmetry property, to each solution defined by the four functions

$$x(t),\ y(t),\ \dot{x}(t),\ \dot{y}(t)$$

corresponds a new solution defined by the four functions

$$x(-t),\ -y(-t),\ -\dot{x}(-t),\ \dot{y}(-t)$$

This is seen by direct substitution in the equations of motion (or also in the Lagrangian). This is the symmetry or mirror image theorem (Fig. 1a). The two arcs of orbit are symmetric

Table 1 Coordinates of the five Lagrange points

	x	y	E_0
L_1	+0.863893	0	−1.59419
L_2	+1.155699	0	−1.58610
L_3	−1.005064	0	−1.506076
L_4	+0.4878495	+0.8660254	−1.493996
L_5	+0.4878495	−0.8660254	−1.493996

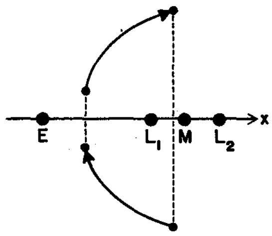

Fig. 1a The mirror image theorem.

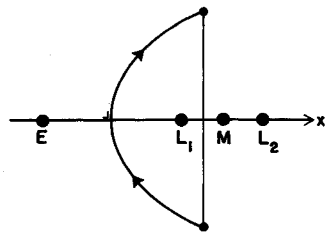

Fig. 1b The mirror image theorem for orbits perpendicular to the syzygy line.

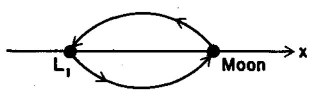

Fig. 1c Trajectory from L_I to the Moon and the symmetric return trajectory.

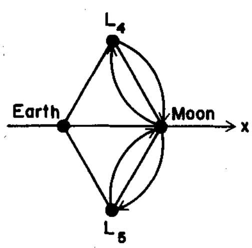

Fig. 1d Symmetric transfers between L_4, L_5, and the Moon.

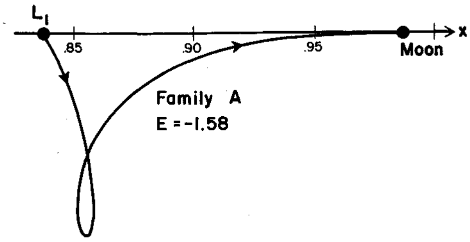

Fig. 2 Family A of orbits from L_I to the Moon ($E = -1.58$).

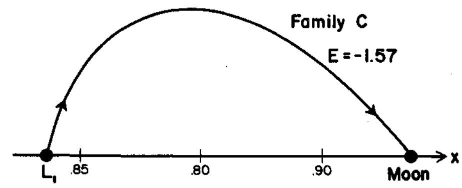

Fig. 3 Branch C of slow orbits from L_I to the Moon ($E = -1.57$)

with respect to the x axis, but the directions of the motion are opposite. In Fig. 1b we illustrate a particular case of the theorem: orbits which cross the syzygy line at a right angle are symmetric with respect to this line. In other words, the original orbit and the mirror image coincide.

In Fig. 1c we show a trajectory from L_I to the Moon and its image, which goes from the Moon back to L_I. The transit times and intersection angles with the x axis are the same for both trajectories. The geometry for L_2 and the Moon is identical. In Fig. 1d we show the corresponding situation for L_4 and L_5. More exactly, we see four trajectories: Moon to L_4 and L_4 to the Moon, together with their respective images which are, in this case, L_5 to the Moon and Moon to L_5.

In the following sections we will describe some numerical results: orbits which connect the Moon with either L_I, L_2, L_4, or L_5. We have not studied the collinear equilibrium point L_3, which is behind the Earth (as seen from the Moon) at about the Earth-Moon distance and which, for this reason, seems to have less astronautical interest than the other four equilibrium points. At any one of these equilibrium points, L_I, we need only two more parameters to determine a trajectory in an unambiguous way. We use an angle α, the angle of the velocity vector with the x axis, and the magnitude V of the residual velocity or the corresponding energy. If the energy E is given, the residual velocity V is computed by

$$\tfrac{1}{2} V^2 = E - E_L$$

where E_L is the potential energy at the particular Lagrange point, as given in Table 1. The initial conditions are thus the coordinates of the Lagrange point together with the two velocity components.

$$\dot{x}_0 = V \cos\alpha = \sqrt{2(E-E_L)}\ \cos\alpha$$

$$\dot{y}_0 = V \sin\alpha = \sqrt{2(E-E_L)}\ \sin\alpha$$

The results which are described in the following sections were obtained by numerical integrations, with systematic variations of the two parameters α and E.

A two-point boundary value problem was solved in regularized variables. We had to find an initial flight path angle to satisfy the given final condition, which is to end at a given point on the Earth-Moon line (the Moon itself). The numerical integrations were stopped at the intersection with this line and, with the use of interpolations, the required trajectory was usually found in about five iterations.

IV. Trajectories from L_I to the Moon

The Lagrangian point L_I is the collinear equilibrium point that is in between the Earth and the Moon, about 60,000 km on this side of the Moon. We found several families of transit orbits from L_I to the Moon which are a reasonable compromise between two characteristics: low energy or residual relative velocity at L_I, as well as short travel time. Only two families are described here. One family (called family A) is a family of slow indirect trajectories, while the other two families contain a branch of fast direct trajectories and a branch of slow trajectories. Each one of these families depends on a single parameter. With the family A of orbits we can leave the L_I-point with as small a velocity as we please (100 m/s for instance) and reach the Moon after a time of about eight days. The trajectories with higher energy and higher velocity at L_I reach the Moon after a longer time. For instance, with a velocity of 435 m/s at L_I (energy -1.50) the angle is 322 deg and the time of flight about fourteen days. This property makes the family A less interesting from the practical point of view.

In Fig. 2 we display a typical orbit of family A, as seen in the rotating coordinate system and corresponding to the energy -1.58.

The family BC of direct orbits has a different property because it does not exist for energies lower than -1.5725 (residual velocity of 210 m/s) (see Fig. 3). The particular orbit with the lowest energy has a transit time of two days. As can be seen on the graph of Fig. 4, this family has two branches; the left branch B corresponds to the fast trajectories (about one day), while the right branch C corresponds to the slow trajectories (about six days).

Let us note here than an example of a transfer trajectory of our present family B was previously given by Farquhar[7] (p. 110). His trajectory has a residual velocity of 570 m/s at L_I and a transit time of 24 h from L_I to the Moon.

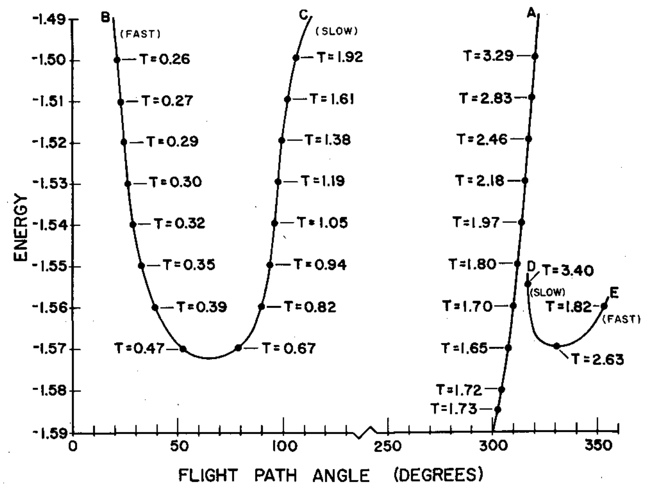

Fig. 4 Initial conditions for orbits from L_1 to the Moon.

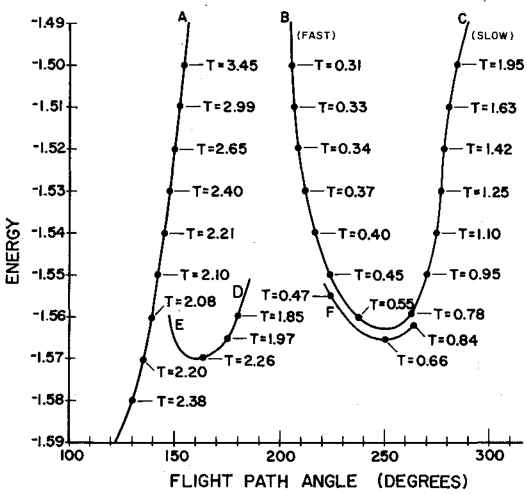

Fig. 5 Initial conditions for orbits from L_2 to the Moon.

V. Trajectories from L_2 to the Moon

The Lagrangian point L_2 is on the Earth-Moon line, about 60,000 km behind the Moon. Four families of trajectories which connect L_2 to the Moon are described. Several other families exist but they are of less practical interest either because of their high energy or the long transfer time.

The family A of indirect trajectories has only one single branch, as is seen in Fig. 5. The energy on this family of orbits goes down as low as the energy of L_2 itself (-1.586), where the residual velocity is then near zero. The travel time on such an orbit is close to ten days. Figure 6 displays an orbit of this family with low energy -1.58 or residual velocity 110 m/s. If the circular sidereal motion around the Earth is considered, it is seen that the satellite stays permanently a few degrees ahead of the Moon and L_2. A trajectory in this family was previously given by D'Amario and Edelbaum[11] (p. 459).

Another family which has been studied in detail is the family BC. It has a branch of fast orbits (B) and a branch of slow orbits (C). The branch B could be called the direct orbits. On the orbits of both branches, the satellite is a few degrees behind L_2 and the Moon, when considered in the fixed coordinate system. The two branches B and C join at a minimum energy point of -1.563 (residual velocity of 210 m/s). The travel time on this orbit is three days.

With a higher energy of -1.50, for instance (420 m/s), the travel times are 1.3 days on the fast branch, while they are eight days on the slow branch. A typical orbit of family BC is shown in Fig. 7 with energy -1.56.

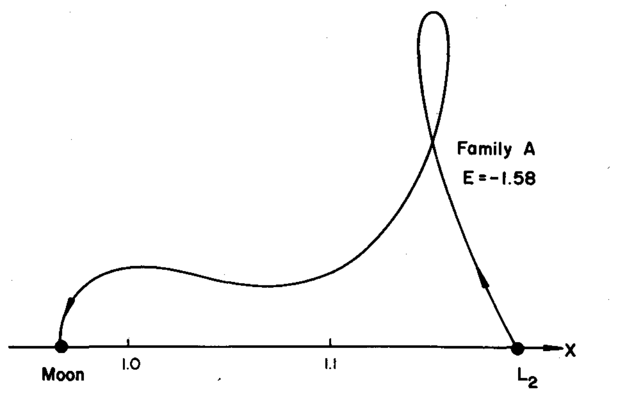

Fig. 6 Low-energy orbit from L_2 to the Moon (family A, $E = -1.58$).

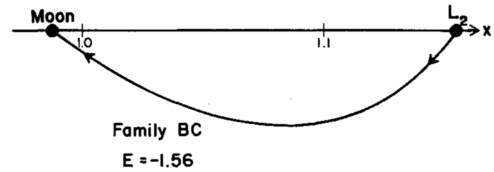

Fig. 7 Low-energy orbit from L_2 to the Moon (family BC, $E = -1.56$).

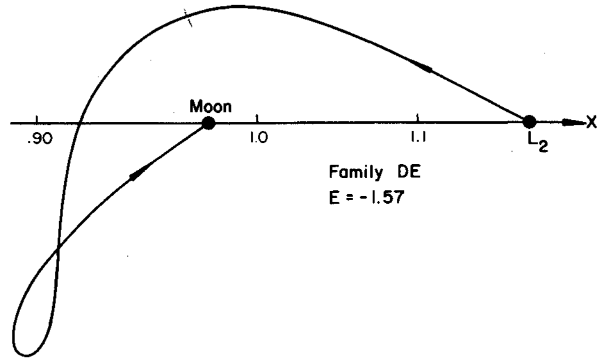

Fig. 8 Family DE of orbits from L_2 to the Moon ($E = -1.57$).

A similar family DE of orbits has also been found with a slow branch and a fast branch. The two branches join at the minimum energy of -1.5705 (equivalent to a residual velocity of 180 m/s). A typical orbit of this family is shown in Fig. 8. It has an energy of -1.57 and a travel time of ten days.

Figure 9 shows another orbit (family F) which was found. It is also a fast trajectory, very similar to the orbits of branch B. The only difference is that the approach arc with the Moon is on the other side of the Moon. The orbit shown has an energy of -1.557 and a travel time of 2.1 days. A few similar fast trajectories are in Ref. 11. (p. 458).

VI. Orbits from L_4 to the Moon

The triangular libration point L_4 is 60 deg ahead of the Moon in its circular motion around the Earth. A single family AB of trajectories from L_4 to the Moon is described here. As in the previous families, we again have a fast branch A and a slow branch B. The two branches connect at the minimum value of the energy, which here is -1.4922 (equivalent to a velocity at L_4 of 60 m/s). The travel time from L_4 to the Moon on the minimum energy trajecctory is 22 days. Figure 10 displays the initial condition curve for the family (energy and travel time vs the flight path angle α *at* L_4). In Fig. 11 we show two typical orbits of this family. One of them (Fig. 11a) is the minimum energy trajectory, while the other is a slow

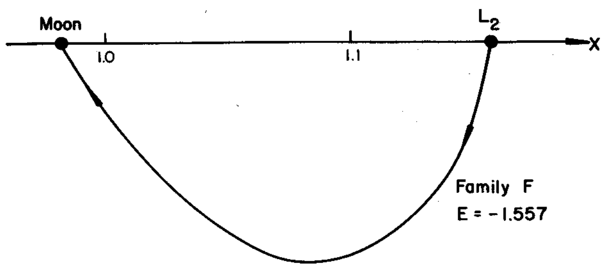

Fig. 9 Family F of orbits from L_2 to the Moon ($E = -1.557$).

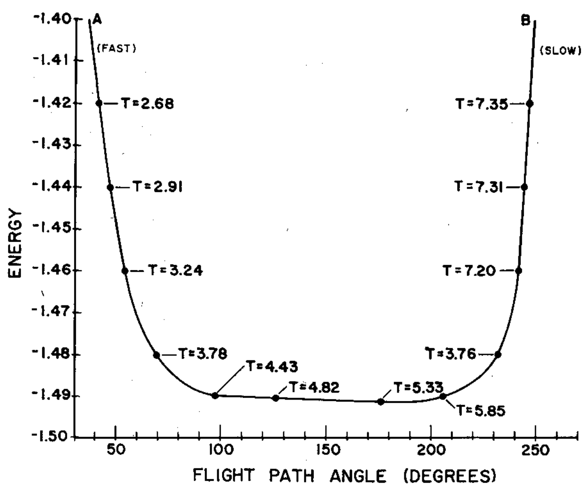

Fig. 10 Initial conditions for orbits from L_4 to the Moon.

trajectory corresponding to $E = 1.45$ (304 m/s). The travel time on the slow trajectory is 31.5 days (Fig. 11b).

On the fast trajectories, the satellite is permanently on the outside of the Moon's circular orbit. On some of the slow trajectories the satellite starts off toward the Earth, but it eventually turns around (to the right, according to the Coriolis forces!), ending up outside the Moon's orbit.

The mirror image orbits of this family are the orbits from the Moon to L_5. They have the same flight times and energy characteristics as the present family.

VII. Orbits from L_5 to the Moon

The triangular point L_5 is 60 deg behind the Moon in its circular motion around the Earth. It turns out that the orbits from L_5 to the Moon form a family which is very similar to the previous one relative to L_4 (Fig. 12). Again, there are two branches (A fast and B slow) which connect at the minimum energy orbit. Here, the minimum energy is about -1.491, which corresponds to the residual velocity or initial velocity at L_5 of 80 m/s. The flight time from L_5 to the Moon is fifteen days. Thus, it is faster to travel from L_5 to the Moon than from L_4 to the Moon, at least on the minimum energy trajectory. The reason for this is easily seen on Fig. 13a, which illustrates one of the orbits. The transfer from L_5 to the Moon is inside the Moon's circular orbit, and the sidereal motion will thus be faster than the Moon's motion. The mirror image orbits are the orbits from the Moon to L_4 (Fig. 13b).

VIII. Relations with the Known Periodic Orbits in the Earth-Moon System

It is well known that a large amount of research has been done in the last century on the study and classification of period orbits of the restricted three-body problem.[19,26-28] Two periodic orbits in the Earth-Moon system which are of special interest in connection with our present problem of traveling from the Moon to L_4 and L_5 are pointed out here. These two

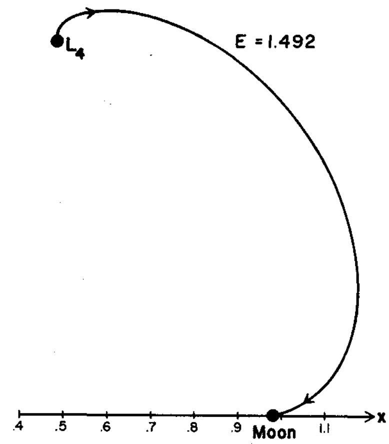

Fig. 11a Minimum-energy trajectory from L_4 to the Moon ($E = -1.492$).

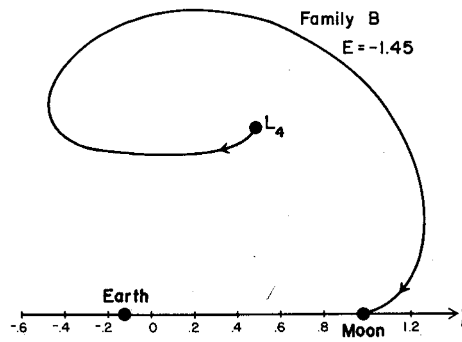

Fig. 11b A slow trajectory from L_4 to the Moon ($E = -1.45$).

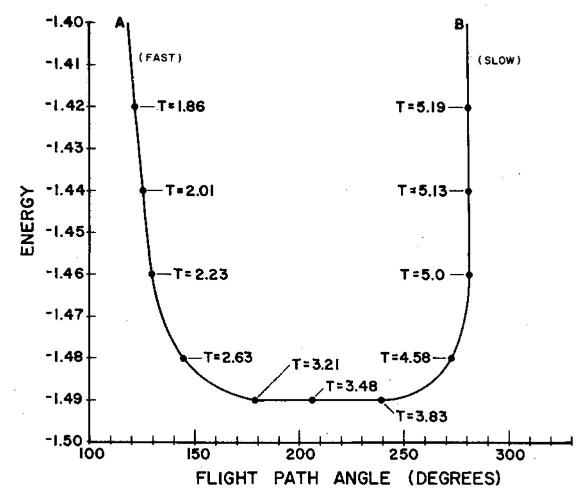

Fig. 12 Initial conditions for orbits from L_5 to the Moon.

orbits are taken from Broucke,[19] and they belong to the families G (around L_1) and I (around L_2). They are shown in Fig. 14. They are also of interest in traveling from L_4 to L_5 or L_5 to L_4.

The first of the two orbits (Fig. 14a) belongs to a family of retrograde orbits around L_1. Within this one-parameter family, one particular orbit can be found which passes through the points L_4 and L_5, while it also passes close to the Moon. The motion is in the direction L_5 to L_4. The orbit crosses the Earth-Moon line orthogonally about halfway in between the Earth and the Moon. The energy is near to -1.40 and the total period is about seven canonical units (30 days).

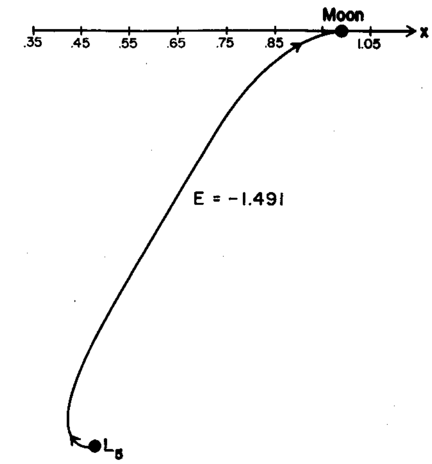

Fig. 13a Minimum-energy trajectory from L_5 to the Moon ($E = -1.491$).

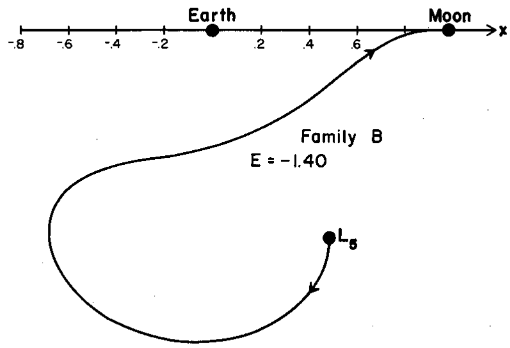

Fig. 13b Slow trajectory from L_5 to the Moon (branch B, $E = -1.40$).

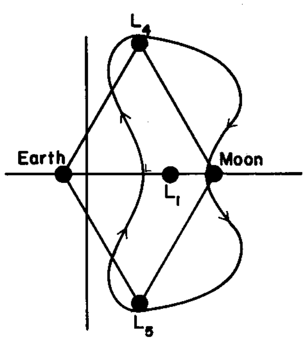

Fig. 14a Periodic orbit around L_1, through L_4 and L_5, passing near the Moon.

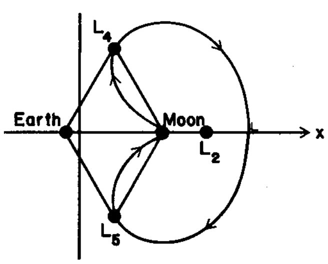

Fig. 14b Periodic orbit around L_2, through L_4 and L_5, passing near the Moon.

The arc L_4-Moon is very similar to some of the previous orbits described in this article.

An orbit with similar properties exists around the libration point L_2 (family I). This periodic orbit (Fig. 14b) also passes through L_4 and L_5 (traveling from L_4 to L_5 this time) and passes very near the Moon. It intersects the Earth-Moon line at a right angle behind the Moon. The energy is -1.43 and the total period is eight canonical units of time (35 days). The arc L_5-Moon is similar in form to one of the trajectories described above.

Our free-fall trajectories can be optimized by applying one or several impulses to the best trajectories that were found. These trajectories would be candidates for the determination of minimum impulse (minimum fuel) trajectories with the methods of D'Amario and Edelbaum,[11] or Jezewski and Rozendaal,[29] as well as for the study of specialized applications, such as Heppenheimer's[30,31] achromatic trajectories.

Acknowledgments

The present research was made possible through a grant from the Bureau of Engineering Research at the University of Texas at Austin. We also want to thank our student, D. Walker, for much help in the programming and computer work related to the present investigation.

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